Bounds for the Rate Distortion Region of `Two-Terminal Common Function Reconstruction' Problem

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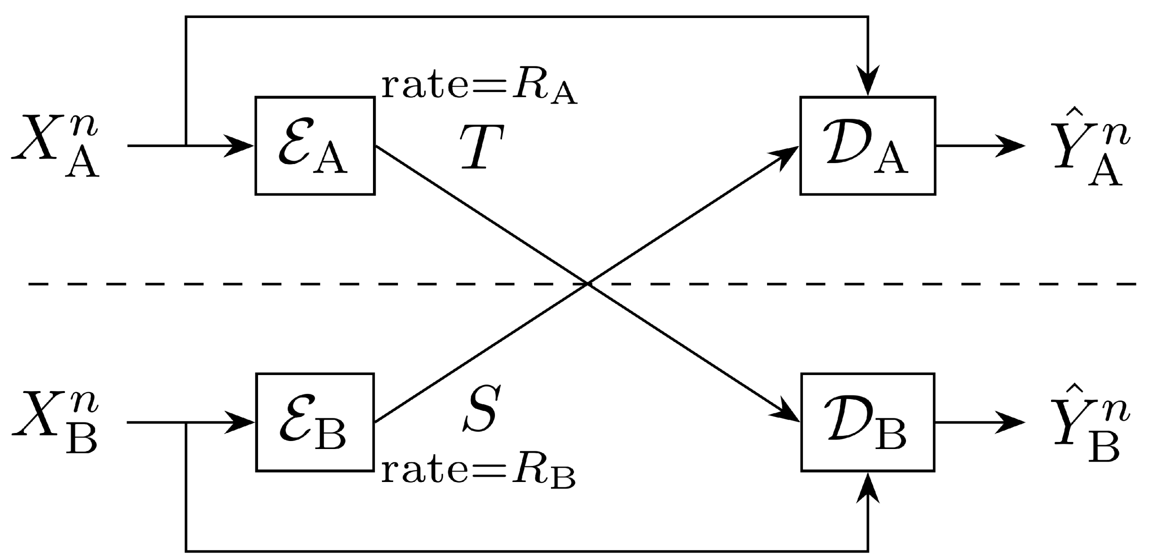
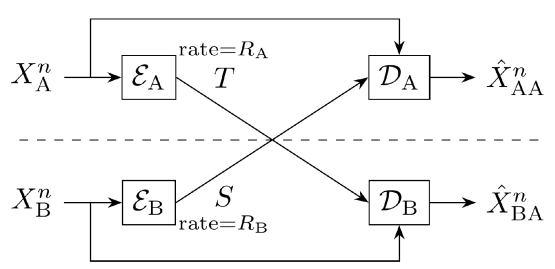
Problem Setup



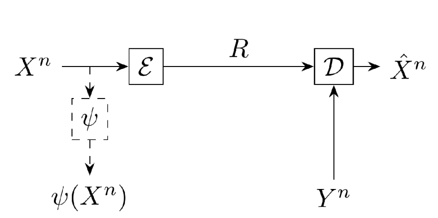
Let $Y^n \in \mathcal{Y}^n$ as $Y_i = g(X_{Ai}, X_{Bi})$.
Want $\mathbb{E}[d(Y^n, \hat{Y}_A^n)] \le D$ and $\mathbb{E}[d(Y^n, \hat{Y}_B^n)] \le D$. Also want $\Pr(\hat{Y}_A^n \ne \hat{Y}_B^n)$ to be small.
Our work: Finding bounds on the set of achievable scalar triples (R_A, R_B, D).

Simple Example: $g(X_A, X_B) = X_A$



Case 1: $R_B \ge H(X_B|X_A)$

- X_B can be recovered at $\mathcal{D}_A$ losslessly.
- Both decoders have access to same information.
- $R_A = \min I(X_A; U|X_B)$ (Wyner-Ziv rate), where min is over $p(u|x_A)p(\hat{x}|u, x_B)$.
- Provably tight.



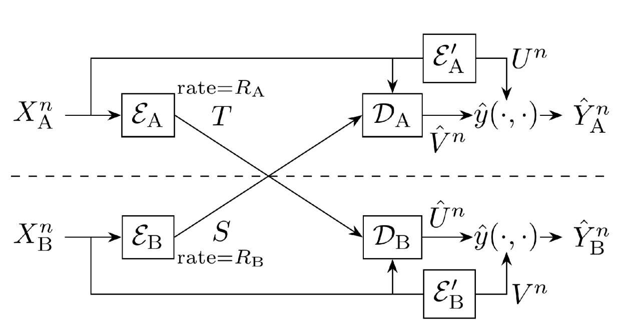
Case 2: $R_B = 0$

- An extension of Steinberg's CR problem[1].
- $R_A = \min I(X_A; U|X_B)$, where min is over $p(u|x_A)$.
- Provably tight.

For $R_B \in (0, H(X_B|X_A))$, can time share between rate regions of case 1 and 2. This result may not be tight.

[1] Yossef Steinberg. "Coding and common reconstruction". In: *IEEE Trans. Inform. Theory* 55.11 (2009), pp. 4995–5010

Inner Bound (Similar to the Berger-Tung Inner Bound)



- Encode X_A^n into U^n, X_B^n into V^n.
- Both decoders recover (w.h.p.) both U^n and V^n.
- Estimate Y^n as $\hat{Y}_A^n$ and $\hat{Y}_B^n$. $\hat{Y}_{Ai} = \hat{y}(U_i, \hat{V}_i)$ and $\hat{Y}_{Bi} = \hat{y}(\hat{U}_i, V_i)$.

A scalar triple (R_A, R_B, D) is CFR-achievable if

$$R_A > I(X_A; U|X_B, Q) \quad \text{and} \quad R_B > I(X_B; V|X_A, Q),$$

for some conditional probability mass function

$$p(q)p(u|x_A, q)p(v|x_B, q)$$

and a function

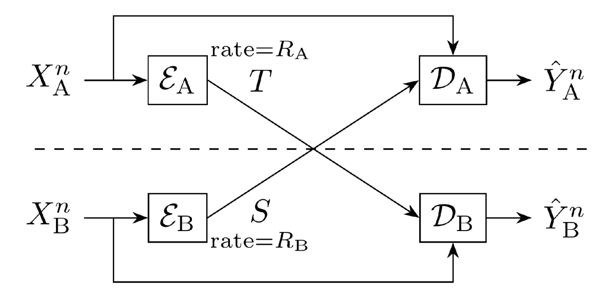
$$\hat{y}(u, v, q)$$

such that for $Y = g(X_A, X_B)$,

$$D \ge \mathbb{E}[d(Y, \hat{Y})].$$

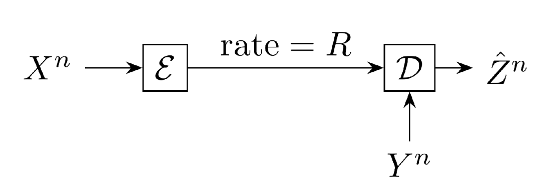
An Easy Outer Bound - Waive the Common Reconstruction Constraint

Waive CR constraint:



- Relaxation yields an outer bound (a converse result).
- CFR problem reduces to *two* source coding with side information problems.
- Rate regions can be determined independently.

Yamamoto's problem[8]**:**



- $\hat{Z}^n$ approximates Z^n where $Z_i = g(X_i, Y_i)$.
- Rate region has been fully determined.

[8] Hirosuke Yamamoto. "Wyner-Ziv theory for a general function of the correlated sources". In: *IEEE Trans. Inform. Theory* 28.5 (1982), pp. 803–807

If an $(n, 2^{n(R_A+\epsilon_n)}, 2^{n(R_B+\epsilon_n)}, D, \epsilon_n)$-CFR coding scheme exists, then for $Y = g(X_A, X_B)$,

$$R_A \ge I(X_A; U|X_B)$$

for some conditional PMF

$$p(u|x_A)$$

and function

$$\hat{y}_B(x_B, u)$$

where

$$U - X_A - X_B$$

forms a Markov chain, and

$$\mathbb{E}[d(Y, \hat{Y}_B)] \le D.$$

$$R_B \ge I(X_B; V|X_A)$$

for some conditional PMF

$$p(v|x_B)$$

and function

$$\hat{y}_A(x_A, v)$$

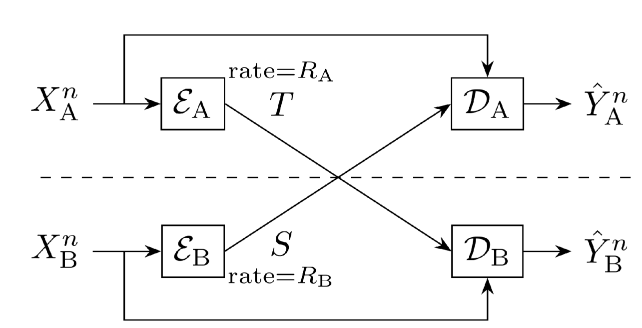
where

$$V - X_B - X_A$$

forms a Markov chain, and

$$\mathbb{E}[d(Y, \hat{Y}_A)] \le D.$$

Tightening the Outer Bound Using Common Reconstruction



Use Fano's inequality

- $\Pr(\hat{Y}_A^n \ne \hat{Y}_B^n) \le \epsilon_n \implies H(\hat{Y}_A^n|\hat{Y}_B^n) \le H(\epsilon_n) + \epsilon_n \log(|g_A| - 1) = n\delta_n$ with $\lim_{n\to\infty} \delta_n = 0$.
- This implies

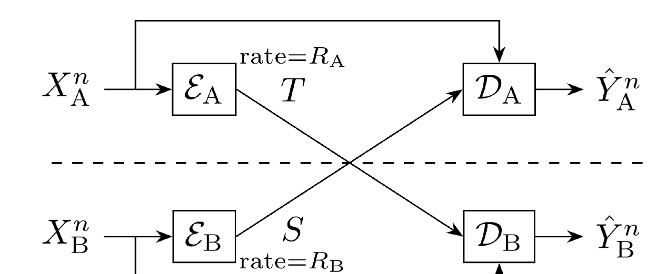
$$nR_A \ge I(X_A^n; \hat{Y}_A^n|X_B^n) - n\delta_n.$$

- For time-sharing random variable Q this implies

$$R_A \ge I(X_A; \hat{Y}_A|X_B, Q) - \delta_n.$$

- Compare with $R_A \ge I(X_A; U|X_B)$ in Yamamoto bound.
- R_A must satisfy both bounds.

Put Together the Findings for a Tighter Outer Bound



If an $(n, 2^{n(R_A+\epsilon_n)}, 2^{n(R_B+\epsilon_n)}, D, \epsilon_n)$-CFR coding scheme exists, then for $Y = g(X_A, X_B)$,

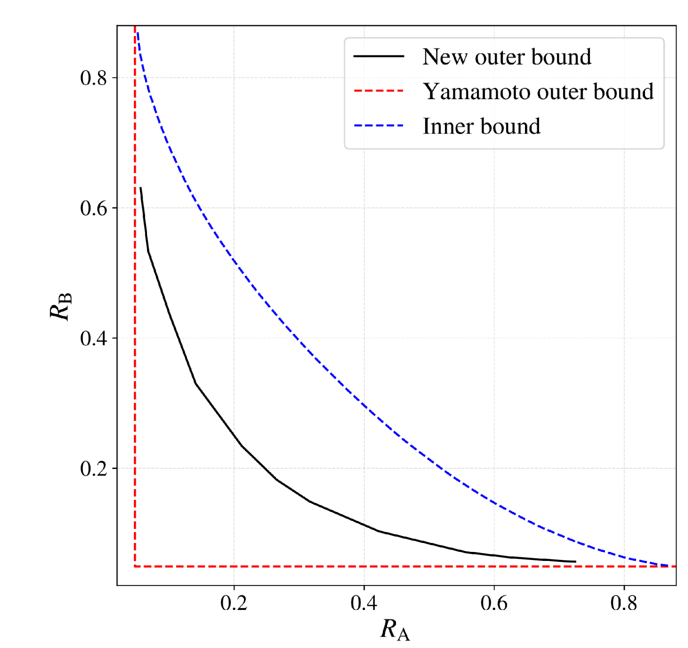
$$R_A \ge \max(I(X_A; U|X_B, Q), I(X_A; \hat{Y}_A|X_B, Q))$$

and

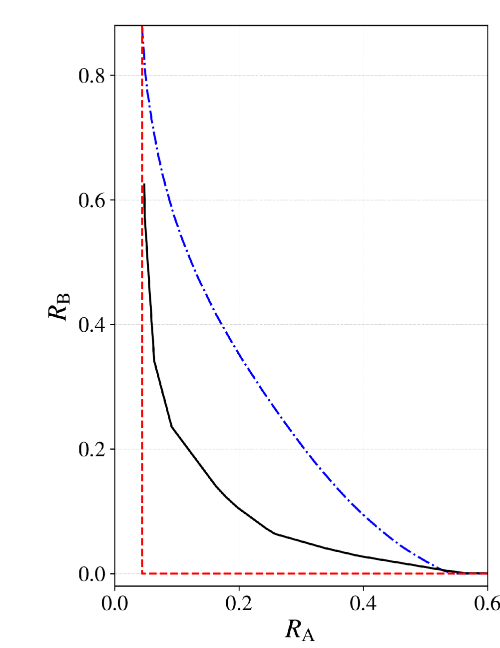
$$R_B \ge \max(I(X_B; V|X_A, Q), I(X_B; \hat{Y}_B|X_A, Q))$$

for some conditional pmf $p(q)p(u, v|x_A, x_B, q)$
and functions $\hat{y}_A(x_A, v, q)$ and $\hat{y}_B(x_B, u, q)$
such that $U - X_A - X_B$ and $X_A - X_B - V$
form two Markov chains, and $\mathbb{E}[d(Y, \hat{Y}_A)] \le D$ and $\mathbb{E}[d(Y, \hat{Y}_B)] \le D$.

Comparison of Bounds for Binary Sources



$(X_A, X_B) \sim \text{DSBS}(0.3)$
$g = \text{AND}$, $D = 0.13$



$X_A \sim \text{Bern}(0.4)$, $Z \sim \text{Bern}(0.3)$,
$X_B = X_A + Z$, $g = \text{AND}$, $D = 0.15$

Next Steps

Open for exploration:

- The parameter space is equivalent in outer bound and Yamamoto bound.
- Only the expressions for the rates are different.
- Can the common reconstruction constraint impose additional structure in the parameter space to match the inner bound?
  e.g., Two chains $U - X_A - X_B$ and $X_A - X_B - V$ are weaker than the condition $U - X_A - X_B - V$.