Two-terminal source coding with common sum reconstruction

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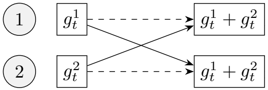
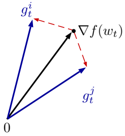
Motivation

Some algorithms require distributed sum (or mean) computation.



Basic building block



Two-terminal stochastic gradient descent (SGD).

Two terminals want to update

$$w_{t+1} = w_t - \eta_t(g_t^i + g_t^j)/2.$$

(1) Need only $g_t^i + g_t^j$, no need to recover g_t^i and g_t^j separately.

(2) g_t^i and g_t^j are correlated.

(3) Two terminals must recover identical sums to (remain) synchronized.

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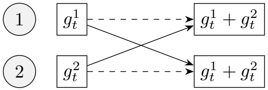

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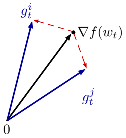

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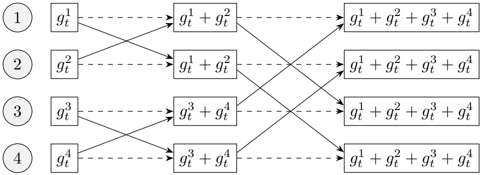
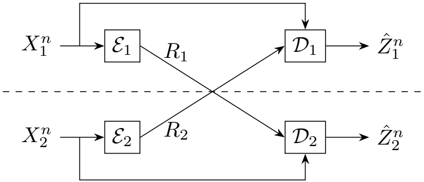
Algorithms exist for extending to more than 2 terminals.



- E.g., Butterfly all-reduce.
- Recursively apply the basic building block.
- 4 terminals take $\log 4 = 2$ rounds for sum computation.
- Other examples: power iteration, k-means.

Two-terminal source coding with "Common Sum Reconstruction" (CSR)

Problem setup:



- Two decoders each want to form two estimates ($\hat{Z}_1^n$ and $\hat{Z}_2^n$) of the sum $Z^n = X_1^n + X_2^n$.
- Want $\Pr(\hat{Z}_1^n \neq \hat{Z}_2^n)$ to be small.
- The decoders may use X_1^n and X_2^n as side information at $\mathcal{D}_1$ and $\mathcal{D}_2$.

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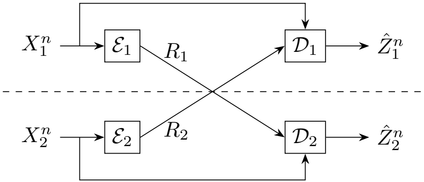

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CSR-achievability:

- A scalar triple (R_1, R_2, D) is CSR-achievable if there exists a coding scheme such that
  - $\mathbb{E}[d(Z^n, \hat{Z}_1^n)] \leq D$ and $\mathbb{E}[d(Z^n, \hat{Z}_2^n)] \leq D$.
  - $\Pr(\hat{Z}_1^n \neq \hat{Z}_2^n)$ can be made arbitrarily small.
- $\mathcal{R}_{\text{CSR}} \subset \mathbb{R}^3$ is the set of all CSR-achievable triples.

Our contributions

Focus on binary/hamming case.

- $X_1, X_2 \in \mathbb{F}_2$, and $d(\cdot,\cdot)$ is Hamming distortion.
- $X_1 + X_2$ is modulo-two sum.
- (X_1, X_2) is a doubly symmetric binary source (DSBS).

X_1 \ X_2	0	1
0	$\frac{(1-p)}{2}$	$\frac{p}{2}$
1	$\frac{p}{2}$	$\frac{(1-p)}{2}$

Joint distribution of a DSBS(p).

Develop two inner bounds to $\mathcal{R}_{\text{CSR}}$ (achievability results).

- Inner bound 1 – Based on Steinberg's common reconstruction (CR) problem[1].
- Inner bound 2 – Based on the lossy version of Körner-Marton's modulo-two sum (LKM) problem[2].

Develop an outer bound to $\mathcal{R}_{\text{CSR}}$ (a converse result).

- Based on Wyner-Ziv's source coding with side information problem[3].

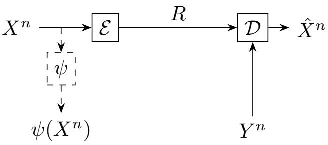
[1] Yossef Steinberg. "Coding and common reconstruction". In: *IEEE Trans. Inform. Theory* 55.11 (2009), pp. 4995–5010

[2] Janos Körner and Katalin Marton. "How to encode the modulo-two sum of binary sources". In: *IEEE Trans. Inform. Theory* 25.2 (1979), pp. 219–221

[3] Aaron Wyner and Jacob Ziv. "The rate-distortion function for source coding with side information at the decoder". In: *IEEE Trans. Inform. Theory* 22.1 (1976), pp. 1–10

Inner bound 1 – Based on Steinberg's CR problem

Steinberg's CR problem



- Similar setup to source coding with side information. Want $\psi(X^n) = \hat{X}^n$ w.h.p. in addition.
- Achievable rate distortion region is known.

$$R_{\mathsf{CR}}(D) = \begin{cases} H(p * D) - H(D) & \text{if } 0 \leq D \leq \frac{1}{2}. \\ 0 & \text{if } \frac{1}{2} \leq D. \end{cases}.$$

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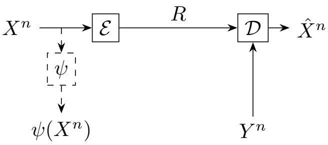

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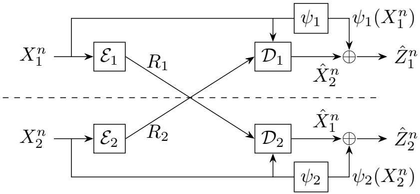
Application to CSR problem



$\hat{Z}_1^n = \psi_1(X_1^n) + \hat{X}_2^n$, $\hat{Z}_2^n = \hat{X}_1^n + \psi_2(X_2^n)$

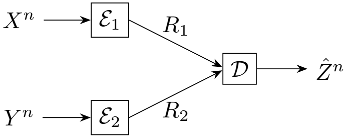
- Employ two parallel Steinberg systems.

$$\mathcal{R}_{\mathsf{A}} = \left\{ (R_1, R_2, D) \;\middle|\; \begin{array}{l} 0 \le D_1, D_2 \le \frac{1}{2}, \\ R_1 \ge R_{\mathsf{CR}}(D_1), \\ R_2 \ge R_{\mathsf{CR}}(D_2), \\ D \ge D_1 * D_2 \end{array} \right\},$$

$$\mathcal{R}_{\mathsf{B}} = \{(R_1, R_2, D) \mid R_1 \ge 0, R_2 \ge 0, D \ge p\}.$$

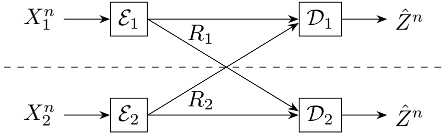
- Then, convex hull $\mathrm{conv}(\mathcal{R}_{\mathsf{A}} \cup \mathcal{R}_{\mathsf{B}}) \subseteq \mathcal{R}_{\mathsf{CSR}}$.

Lossy Körner-Marton problem



$\mathcal{D}$ estimates the sum $Z^n = X^n + Y^n$ as $\hat{Z}^n$.

Application of LKM to CSR problem

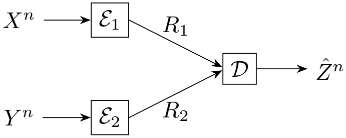


Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are identical LKM decoders.

[4] Sung Hoon Lim et al. "Towards an Algebraic Network Information Theory: Distributed Lossy Computation of Linear Functions". In: *Proc. Int. Symp. Inform. Theory*. IEEE. 2019, pp. 1827–1831

Inner bound 2 – Based on Lossy Körner-Marton (LKM) problem

Lossy Körner-Marton problem



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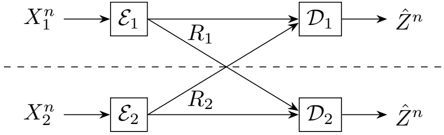


Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are identical LKM decoders.

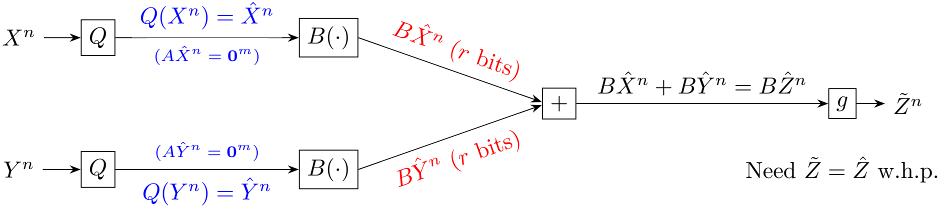
- A scalar triple (R_1, R_2, D) is LKM-achievable if there exists a coding scheme such that $\mathbb{E}[d(Z^n, \hat{Z}^n)] \leq D$.
- $\mathcal{R}_{\text{LKM}} \subset \mathbb{R}^3$ is the set of all LKM-achievable triples. Note $\mathcal{R}_{\text{LKM}} \subseteq \mathcal{R}_{\text{CSR}}$.
- Lossless version ($D = 0$): Tight bound is known (Körner and Marton 1979).
- Lossy version ($D > 0$): Joint typicality-based encoding/decoding to obtain an inner bound[4].

[4] Sung Hoon Lim et al. "Towards an Algebraic Network Information Theory: Distributed Lossy Computation of Linear Functions". In: *Proc. Int. Symp. Inform. Theory*. IEEE. 2019, pp. 1827–1831

Claim:
All triples in $\{(R_1, R_2, D) \mid R_1, R_2 \geq H(p * q * q) - H(q), D \geq q * q, 0 \leq q \leq \frac{1}{2}\}$ are LKM-achievable.

Proof sketch:

- $U \in \mathbb{F}_2^{m \times n}$: parity check matrix of a 'good' linear source code with avg. distortion q.
- $V \in \mathbb{F}_2^{r \times n}$: parity check matrix of a 'good' linear channel code for BSC($p * q * q$).



An achievability result for LKM problem

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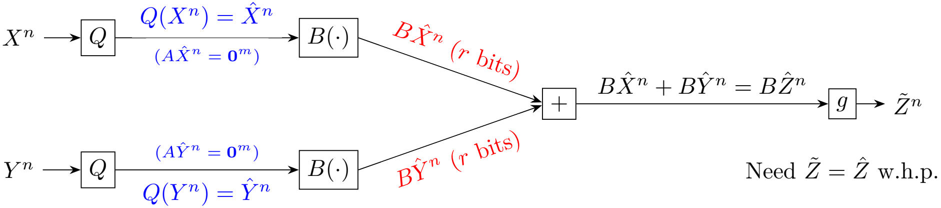
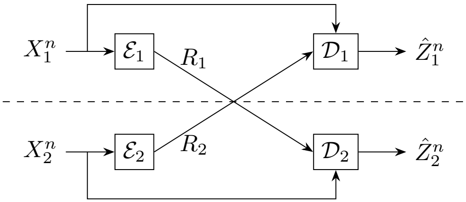


- Can show that we only need $r > H(p*q*q) - H(q)$.
- Resulting average distortion $\mathbb{E}[d(Z, \hat{Z})] \leq q*q$. (Recall $Z = X + Y$, $\hat{Z}^n = \hat{X}^n + \hat{Y}^n$.)

Outer bound with Wyner-Ziv problem

Waive the common reconstruction (CR) constraint in CSR

- CSR problem reduces to *two* source coding with side information (Wyner-Ziv) problems.
- This relaxation yields an outer bound (a converse result).



$\hat{Z}_1^n = X_1^n + \hat{X}_2^n$ (use X_1^n as side info. to decode $\hat{X}_2^n$).

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Outer bound with Wyner-Ziv problem

Waive the common reconstruction (CR) constraint in CSR

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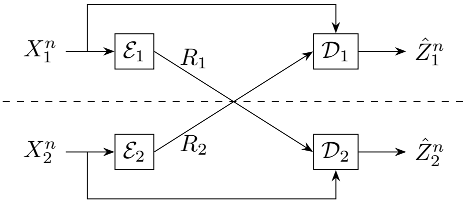

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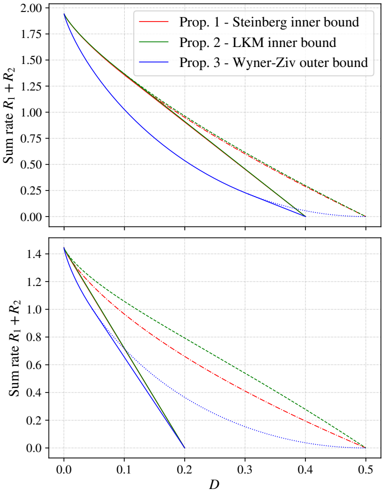
- $R_{\mathsf{WZ}}(D) = \text{l.c.e}(g(D))$, the lower convex envelope of $g(D) = \begin{cases} H(p * D) - H(D) & \text{if } 0 \le D < p, \\ 0 & \text{if } p \le D. \end{cases}$
- $\mathcal{R}_{\mathsf{CSR}} \subseteq \{(R_1, R_2, D) \mid R_1, R_2 \ge R_{\mathsf{WZ}}(D), D \ge 0\}$.

Comparison of bounds

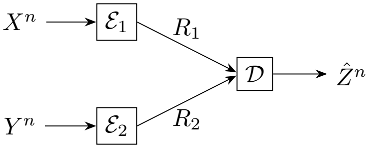
Special case $R_1 = R_2$:

- (X_1, X_2) is a DSBS(p) with $p = 0.4$ (top) and $p = 0.2$ (bottom).
- The straight portion of the solid lines (i.e., the large D portion) is tangent to the corresponding non-solid lines. This results from time-sharing with the zero-sum-rate point $(p, 0)$.
- The inner and outer bounds get tighter as p decreases.
- Steinberg inner bound is better than LKM inner bound.



Open for exploration:

- Find a distribution for (X_1, X_2) where LKM inner bound outperforms Steinberg inner bound.
- Improve inner bounds by correlating the quantization errors.



$$\begin{aligned}\hat{Z}^n &= \hat{X}^n + \hat{Y}^n \\ &= (X^n + W_X^n) + (Y^n + W_Y^n) \\ &= X^n + Y^n + (\underbrace{W_X^n + W_Y^n}_{\text{correlate to cancel}}).\end{aligned}$$

Conclusion and next steps

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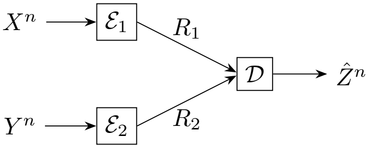

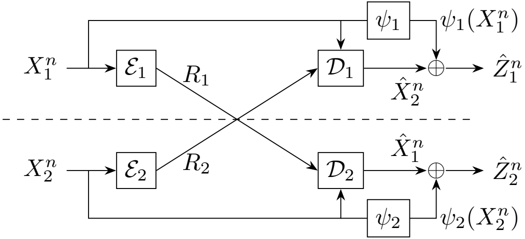

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\hat{Z}_1^n &= \psi_1(X_1^n) + \hat{X}_2^n \\
&= \hat{X}_1^n + \hat{X}_2^n \qquad \text{(w.h.p)} \\
&= X_1^n + X_2^n + (\underbrace{W_1^n + W_2^n}_{\text{correlate to cancel}}).
\end{aligned}
$$

Thank you!.